Relative derivations, classification problems, and PDEs

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Outline

1. Derivations (review)
2. Example 1: extremal Kähler surfaces
3. Example 2: Riemann surfaces with $|\nabla K| = 1$
4. Relative derivations
5. Riemann surfaces with $|\nabla K| = 1$ revisited
6. Prolongations and relative Lie algebroids
7. PDEs and relative Lie algebroids

Main references:

- R.L. Fernandes & Ivan Struchiner, The Global Solutions to a Cartan's Realization Problem, *Mem. Amer. Math. Soc.* **306** (2025), no. 1548.
- R.L. Fernandes & Wilmer Smilde, Relative Lie algebroids and Cartan realization problems, `arXiv:2503.19233`. To appear in *Selecta Mathematica*.

Derivations of a vector bundle

Notations:

- $A \to M$ – vector bundle
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For a vector bundle map

$$\begin{tikzcd} A_1 \arrow[r, "\Phi"] \arrow[d] & A_2 \arrow[d] \\ M_1 \arrow[r, "\phi"'] & M_2 \end{tikzcd}$$

There is an induced pullback map

$$\Phi^* : \Omega^\bullet(A_2) \to \Omega^\bullet(A_1).$$

Remark: $\Omega^0(A) := C^\infty(M)$.

Derivations of a vector bundle

Definition

A k-**derivation** of $A \to M$ is a linear map $\mathrm{D} : \Omega^{\bullet}(A) \to \Omega^{\bullet+k}(A)$ satisfying Leibniz:

$$\mathrm{D}(\alpha \wedge \beta) = (\mathrm{D}\alpha) \wedge \beta + (-1)^{|\alpha|k} \alpha \wedge (\mathrm{D}\beta).$$

The **symbol** of D is the bundle map $\sigma(D) : T^*M \to \wedge^k A^*$, $\sigma(D)(\mathrm{d}f) := \mathrm{D}(f)$.

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- A k-derivation is completely determined by what it does to degree 0 (i.e., functions) and degree 1 forms.
- A morphism $\Phi : (A_1, \mathrm{D}_1) \to (A_2, \mathrm{D}_2)$ is a vector bundle map such that

$$\Phi^* \mathrm{D}_2 \alpha = \mathrm{D}_1 \Phi^* \alpha.$$

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Definition

A k-**bracket** on $A \to M$ is a skew-symmetric, $\mathbb{R}$-multilinear, map

$$[\cdot, \dots, \cdot] : \underbrace{\Gamma(A) \times \cdots \times \Gamma(A)}_{(k+1)\text{-times}} \to \Gamma(A)$$

together with an **anchor** $\rho : \wedge^k A \to TM$ satisfying Leibniz:

$$[\alpha_0, \cdots, f\alpha_k] = f[\alpha_0, \cdots, \alpha_k] + \rho(\alpha_0, \dots, \alpha_{k-1})(f)\alpha_k.$$

Derivations of a vector bundle

Proposition

For a vector bundle $A \to M$ there is a natural 1:1 correspondence

$$\left\{k\text{-derivations}\right\} \stackrel{1:1}{\longleftrightarrow} \left\{k\text{-brackets}\right\}$$

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Proof.

Koszul formula:

$$\langle \sigma(\mathrm{D})(\mathrm{d}f), v_1 \wedge \cdots \wedge v_k \rangle = \langle \mathrm{d}f, \rho(v_1, \ldots, v_k) \rangle$$

$$\mathrm{D}\alpha(v_0, \ldots, v_k) = \sum_{i=0}^{k} (-1)^i \rho(v_0, \ldots, \widehat{v_i}, \ldots, v_k)(\alpha(v_i)) - \alpha([v_0, \ldots, v_k]),$$

$\square$

Derivations of a vector bundle

Denoting $\mathrm{Der}^k(A)$ the space of k-derivations:

(i) $\mathrm{Der}^k(A) = \Gamma(\mathcal{D}^k_A)$ for a vector bundle $\mathcal{D}^k_A \to M$;

(ii) There is a symbol short exact sequence:

$$0 \longrightarrow \mathrm{Hom}(\wedge^{k+1}A, A) \longrightarrow \mathcal{D}^k_A \xrightarrow{\sigma} \mathrm{Hom}(\wedge^k A, TM) \longrightarrow 0.$$

(iii) Graded commutator $[\cdot,\cdot] : \mathrm{Der}^k(A) \times \mathrm{Der}^l(A) \to \mathrm{Der}^{k+l}(A)$ is a graded Lie bracket.

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For a Lie algebroid $A \Rightarrow M$ one gets two complexes:

- **de Rham complex** $C^\bullet(A) := \Omega^\bullet(A)$, $\mathrm{d}_{\mathrm{dR}} := D$.
- **deformation complex** $C^\bullet_{\mathrm{def}}(A) := \mathrm{Der}^{\bullet-1}(A)$, $\mathrm{d}_{\mathrm{def}} := [D, \cdot]$.

Derivations of a vector bundle

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Definition

A **realization** (P, θ, r) of a Lie algebroid (A, D) is a morphism of Lie algebroids

$$\begin{tikzcd} TP \arrow[r, "\theta"] \arrow[d] & A \arrow[d] \\ P \arrow[r, "r"'] & M \end{tikzcd} \qquad \theta^* \circ \mathrm{D} = \mathrm{d} \circ \theta^*.$$

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Proposition

If $\mathcal{G} \rightrightarrows M$ integrates $A \Rightarrow M$ then $(s^{-1}(x), \omega_{\mathrm{MC}}, t)$ is a realization.

An example: extremal Kähler surfaces

A Kähler surface (S, g, Ω, J) is called **extremal** if the Hamiltonian vector field X_K associated with the Gaussian curvature K is an infinitesimal isometry.

Classification Problem

Find all extremal Kähler surfaces up to isomorphism.

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Find all extremal Kähler surfaces up to isomorphism.

Unitary frame bundle:

$$P := \left\{ u : \mathbb{C} \to (T_x S, J_x) \mid \text{complex linear isomorphism} \right\}$$

Tautological form: $\theta \in \Omega^1(P, \mathbb{C})$

Levi-Civita connection: $\omega \in \Omega^1(P, \mathfrak{u}(1))$

Structure Equations: $\begin{cases} \mathrm{d}\theta = -\omega \wedge \theta \\ \mathrm{d}\omega = \frac{K}{2}\, \theta \wedge \bar{\theta} \end{cases}$ (identifying $\mathfrak{u}(1) \simeq i\mathbb{R}$)

An example: extremal Kähler surfaces

- Differentiating K:

$$\mathrm{d}K = -(\bar{T}\theta + T\bar{\theta}), \text{ with } T : P \to \mathbb{C},\ T := \tfrac{i}{2}\theta(\widetilde{X}_K),$$

where $\widetilde{X}_K$ denotes the horizontal lift of X_K;

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$$\mathrm{d}T = U\theta - T\omega, \text{ with } U : P \to \mathbb{R},\ U := \tfrac{i}{2}\omega(\widetilde{X}_K);$$

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- Differentiating T:

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- Differentiating U:

$$\mathrm{d}U = -\tfrac{K}{2}(\bar{T}\theta + T\bar{\theta}).$$

An example: extremal Kähler surfaces

Conclusion: An extremal Kähler surface amounts to:

- $U(1)$-principal bundle $P \to S$ with a coframe $(\theta, \omega) \in \Omega^1(P, \mathbb{C} \oplus i\mathbb{R})$ and
- a map $(K, T, U) : P \to \mathbb{R} \times \mathbb{C} \times \mathbb{R}$

satisfying:

$$\begin{cases} \mathrm{d}\theta = -\omega \wedge \theta \\ \mathrm{d}\omega = \frac{K}{2}\,\theta \wedge \bar{\theta} \\ \mathrm{d}K = -(\bar{T}\theta + T\bar{\theta}) \\ \mathrm{d}T = U\theta - T\omega \\ \mathrm{d}U = -\frac{K}{2}(\bar{T}\theta + T\bar{\theta}) \end{cases} \qquad (\star)$$

The quotient $S = P/U(1)$ is the desired extremal Kähler surface.

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System $(\star)$ defines a degree 1 derivation D satisfying $\mathrm{D}^2 = 0$, i.e., a Lie algebroid!

Algebroid of extremal Kähler surfaces

Vector bundle:

$$A = (\mathbb{R} \times \mathbb{C} \times \mathbb{R}) \times (\mathbb{C} \oplus i\mathbb{R}) \longrightarrow M = \mathbb{R} \times \mathbb{C} \times \mathbb{R}$$

(with global coframe $(\theta, \omega) \in \Omega^1(A)$) (with global coordinates (K, T, U))

with derivation $\mathrm{D} : \Omega^\bullet(A) \to \Omega^{\bullet+1}(A)$ defined by:

$$\begin{cases} \mathrm{D}\theta = -\omega \wedge \theta \\ \mathrm{D}\omega = \frac{K}{2}\, \theta \wedge \bar\theta \\ \mathrm{D}K = -(\bar T\theta + T\bar\theta) \\ \mathrm{D}T = U\theta - T\omega \\ \mathrm{D}U = -\frac{K}{2}(\bar T\theta + T\bar\theta) \end{cases}$$

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Solutions: $U(2)$-realizations of $A\Rightarrow M$ give the unitary frame bundles of extremal Kähler surfaces.

Example: Riemann surfaces with $|\nabla K| = 1$

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Orthogonal frame bundle:

$$P := \left\{ u : \mathbb{R}^2 \to (T_x S, g_x) \mid \text{linear isometry} \right\}$$

Tautological form: $\theta = (\theta^1, \theta^2) \in \Omega^1(P, \mathbb{R}^2)$

Levi-Civita connection: $\omega = \theta^3 \in \Omega^1(P, \mathfrak{so}(2))$

Structure Equations:
$$\begin{cases} \mathrm{d}\theta^1 = -\theta^3 \wedge \theta^1, \\ \mathrm{d}\theta^2 = \theta^3 \wedge \theta^1, \\ \mathrm{d}\theta^3 = K\theta^1 \wedge \theta^2, \end{cases} \qquad (\text{identifying } \mathfrak{so}(2) \simeq \mathbb{R})$$

Example: Riemann surfaces with $|\nabla K| = 1$

- Since $\mathrm{d}K = K_1\theta^1 + K_2\theta^2$, for some functions K_1 and K_2, and $|\nabla K| = 1$:

$$\mathrm{d}K = \cos(\varphi)\theta^1 + \sin(\varphi)\theta^2,$$

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for a new variable c_1;
- Using $\mathrm{d}^2c_0 = 0$, one finds:
$$\mathrm{d}c_1 = -2c_0c_1\mathrm{d}K + c_2(-\sin(\varphi)\theta^1 + \cos(\varphi)\theta^2),$$
for yet a new variable c_2.

Never ends! Produces a sequence of new functions $\varphi, c_0, c_1, \ldots, c_k, \ldots$

Relative derivations

Notations:

- $A \to N$ – vector bundle
- $p : M \to N$ – submersion
- $B := p^*A \to M$ – pullback bundle

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together with an **relative anchor** $\rho : \wedge^k B \to p^*TN$ satisfying Leibniz:

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Proof.

Similar, using modified Koszul formula:

$$(\mathrm{D}f)(p^*v_1, \dots, p^*v_k) = \rho(p^*v_1, \dots, p^*v_k)(f),$$

$$(\mathrm{D}\alpha)(p^*v_0, \dots, p^*v_k) = \sum_{i=0}^{k}(-1)^{i+k}\rho(p^*v_0, \dots, \widehat{p^*v_i}, \dots, p^*v_k)\,(\langle\alpha, v_i\rangle) - p^*\alpha([v_0, \dots, v_k])$$

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But there are some issues...

- No graded Lie bracket on the space of relative derivations!
- D^2 does not make sense!

Relative algebroid

Definition

A **relative algebroid** is a triple (A, p, D) where $A \to N$ is a vector bundle, $p : M \to N$ is a submersion, and D is a relative 1-derivation.

$$\begin{tikzcd} B := p^*A \arrow[r] \arrow[d] & A \arrow[d] \arrow[l, bend left, dashed, "\mathrm{D}"] \\ M \arrow[r, "p"'] & N \end{tikzcd} \qquad \mathrm{D} : \Omega^{\bullet}(A) \to \Omega^{\bullet+1}(B).$$

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- A more precise name would be relative almost Lie algebroid.
- There is a more general notion where $p : M \to N$ is replaced by $(M, \mathcal{F})$.
- D^2 does not make sense, so there seems to be no notion of relative Lie algebroid.

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Definition

A **realization** of a relative algebroid (A, p, D) consists of a manifold P together with a morphism of relative Lie algebroids

$$\begin{tikzcd} TP \arrow[r, "\theta"] \arrow[d] & B \arrow[d] \\ P \arrow[r, "r"'] & M \end{tikzcd} \qquad \theta^* \circ D = \mathrm{d} \circ \theta^*.$$

Riemann surfaces with $|\nabla K| = 1$ revisited

Classification Problem: Find all Riemann surfaces (M, g) whose Gauss curvature K satisfies $|\nabla K| = 1$.

We saw this is described by

$$\begin{cases} \mathrm{d}\theta^1 = -\theta^3 \wedge \theta^1, \\ \mathrm{d}\theta^2 = \theta^3 \wedge \theta^1, \\ \mathrm{d}\theta^3 = K\theta^1 \wedge \theta^2, \\ \mathrm{d}K = \cos(\varphi)\theta^1 + \sin(\varphi)\theta^2. \end{cases}$$

Riemann surfaces with $|\nabla K| = 1$ revisited

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This can be described by a relative derivation D!

Riemann surfaces with $|\nabla K| = 1$ revisited

$$
\begin{tikzcd}
B = p^*A \arrow[r] \arrow[d] & A = \mathbb{R}\times\mathbb{R}^3 \arrow[d] \arrow[l, dashed, bend left, "\mathrm{D}"] \\
M = \mathbb{R}\times\mathbb{S}^1 \arrow[r, "p"'] & N = \mathbb{R}
\end{tikzcd}
$$

(global coframe $\{\theta^1, \theta^2, \theta^3\}$)

(coordinates (K,ϕ))

(coordinate K)

$$
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\mathrm{D}\theta^1 = -\theta^3\wedge\theta^1, \\
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\begin{tikzcd}
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\end{tikzcd}
$$

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$$

What about D^2?

Prolongations

Definition

A **prolongation** of a relative algebroid (A, p, D) is a relative algebroid (B, p_1, D_1) relative to a submersion $p_1\ M_1 \to M$ satisfying

1. (Extension) $\mathrm{D}_1\alpha = p_1^*\mathrm{D}\alpha$ for $\alpha \in \Omega^\bullet(A)$, and
2. (Completion) $\mathrm{D}_1 \circ \mathrm{D} = 0$.

$$\begin{tikzcd}
B_1 = p_1^* B \arrow[r] \arrow[d] & B = p^* A \arrow[r] \arrow[d] \arrow[l, bend left, dashed, "\mathrm{D}_1"] & A \arrow[d] \arrow[l, bend left, dashed, "\mathrm{D}"] \\
M_1 \arrow[r, "p_1"'] & M \arrow[r, "p"'] & N.
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\end{tikzcd}$$

Do prolongations exist? How can one construct them?

Recipe to construct prolongations

$$
\begin{tikzcd}
B_1 = p_1^* B \arrow[r] \arrow[d] & B = p^* A \arrow[r] \arrow[d] \arrow[l, dashed, bend left, "\mathrm{D}_1"] & A \arrow[d] \arrow[l, dashed, bend left, "\mathrm{D}"] \\
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Step 1 The surjective bundle map $\Pi : \mathcal{D}^1_B \to \mathcal{D}^1_p$, $\mathrm{D} \mapsto \mathrm{D} \circ p^*$, gives rise to a space of **pointwise lifts** or **completions** of D:

$$L := \left\{ \tilde{\mathrm{D}}_m \in \mathcal{D}^1_B \mid \Pi(\tilde{\mathrm{D}}_m) = \mathrm{D}_m, \text{ for some } m \in M \right\},$$
$$p_1 : L \to M,\ \tilde{\mathrm{D}}_m \mapsto m.$$

Recipe to construct prolongations

$$\begin{tikzcd}
B_1 = p_1^* B \arrow[rr] \arrow[d] & & B = p^* A \arrow[rr] \arrow[d] \arrow[ll, dashed, bend left, "\mathrm{D}_1"] & & A \arrow[d] \arrow[ll, dashed, bend left, "\mathrm{D}"] \\
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Step 2 For $\tilde{\mathrm{D}}_m \in L$, the composition $\tilde{\mathrm{D}}_m \circ \mathrm{D} : \Omega^\bullet(A) \to B_m$ makes sense. Then

$$M_1 := \left\{ \tilde{\mathrm{D}}_m \in L \mid \tilde{\mathrm{D}}_m \circ \mathrm{D} = 0, \forall m \in M \right\}.$$

Recipe to construct prolongations

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Step 3 $\mathrm{D}_1 : \Omega^\bullet(B) \to \Omega^{\bullet+1}(B_1)$ is defined by:

$$(D_1\alpha)|_{\tilde{\mathrm{D}}_m} := \tilde{\mathrm{D}}_m \alpha.$$

Recipe to construct prolongations

$$\begin{tikzcd} B_1 = p_1^* B \arrow[r] \arrow[d] & B = p^* A \arrow[r] \arrow[d] \arrow[l, dashed, bend left, "\mathrm{D}_1"] & A \arrow[d] \arrow[l, dashed, bend left, "\mathrm{D}"] \\ M_1 \arrow[r, "p_1"'] & M \arrow[r, "p"'] & N. \end{tikzcd}$$

Theorem

The previous prolongation is universal: if a prolongation exists, it factors through the previous one.

- The obstructions to existence can be expressed in terms of tableaux and Spencer cohomology;
- One defines higher order prolongations by iteration;
- One says that a relative algebroid is formally integrable is all prolongations exist.

Prolongations

Definition

A **relative Lie algebroid** is a formally integrable relative algebroid.

$$
\begin{tikzcd}
(B_\infty, \mathrm{D}_\infty)\ \dots \arrow[r] \arrow[d] & B_k \arrow[r] \arrow[d] & B_{k-1} \arrow[r] \arrow[d] \arrow[l, dashed, bend left, "\mathrm{D}_k"] & \dots \arrow[r] & B_1 \arrow[r] \arrow[d] & B \arrow[d] \arrow[l, dashed, bend left, "\mathrm{D}_1"] \\
M_\infty\ \dots \arrow[r] & M_k \arrow[r, "p_k"'] & M_{k-1} \arrow[r] & \dots \arrow[r] & M_1 \arrow[r, "p_1"'] & M
\end{tikzcd}
$$

The limit $(B_\infty, \mathrm{D}_\infty)$ is a profinite dimensional Lie algebroid!

Riemann surfaces with $|\nabla K| = 1$ revisited (once again!)

$$\begin{tikzcd}
B = p^*A \arrow[r] \arrow[d] & A = \underline{\mathbb{R}^3} \arrow[l, dashed, bend left, "\mathrm{D}"] \arrow[d] \\
\mathbb{R} \times \mathbb{S}^1 \arrow[r, "p"'] & \mathbb{R}
\end{tikzcd}
\qquad
\begin{cases}
\mathrm{D}\theta^1 = -\theta^3 \wedge \theta^1, \\
\mathrm{D}\theta^2 = \theta^3 \wedge \theta^1, \\
\mathrm{D}\theta^3 = K\theta^1 \wedge \theta^2, \\
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\end{cases}$$

Riemann surfaces with $|\nabla K| = 1$ revisited (once again!)

$$\begin{tikzcd} B = p^*A \arrow[r] \arrow[d] & A = \underline{\mathbb{R}^3} \arrow[l, dashed, bend left, "\mathrm{D}"] \arrow[d] \\ \mathbb{R} \times \mathbb{S}^1 \arrow[r, "p"'] & \mathbb{R} \end{tikzcd} \qquad \begin{cases} \mathrm{D}\theta^1 = -\theta^3 \wedge \theta^1, \\ \mathrm{D}\theta^2 = \theta^3 \wedge \theta^1, \\ \mathrm{D}\theta^3 = K\theta^1 \wedge \theta^2, \\ \mathrm{D}K = \cos(\varphi)\theta^1 + \sin(\varphi)\theta^2. \end{cases}$$

$$\begin{tikzcd} \underline{\mathbb{R}^3} \arrow[r] \arrow[d] & \cdots \arrow[r] & \underline{\mathbb{R}^3} \arrow[r] \arrow[d] & \underline{\mathbb{R}^3} \arrow[r] \arrow[d] & \cdots \arrow[r] & \underline{\mathbb{R}^3} \arrow[d] \\ \mathbb{R}^\infty \times \mathbb{S}^1 \times \mathbb{R} \arrow[r] & \cdots \arrow[r] & \mathbb{R}^k \times \mathbb{S}^1 \times \mathbb{R} \arrow[r, "p_k"'] & \mathbb{R}^{k-1} \times \mathbb{S}^1 \times \mathbb{R} \arrow[r] & \cdots \arrow[r, "p_1"'] & \mathbb{R}, \end{tikzcd}$$

where $\mathbb{R}^k \times \mathbb{S}^1 \times \mathbb{R}$ has coordinates $(c_{k-1}, \dots, c_0, \varphi, K)$. Prolongation is determined by:

$$\begin{cases} \mathrm{D}\varphi = \theta^3 + c_0\left(-\sin(\varphi)\theta^1 + \cos(\varphi)\theta^2\right) \\ \mathrm{D}c_k = P_k[c_0, \dots, c_k]\mathrm{d}K + c_{k+1}\left(-\sin(\varphi)\theta^1 + \cos(\varphi)\theta^2\right) \end{cases}$$

PDEs and relative Lie algebroids

Example. Consider the PDE for $u = u(x, y)$:

$$\boxed{u_x = y}$$

– Setting $\pi : \mathbb{R}^3 \to \mathbb{R}^2$, $(x, y, u) \mapsto (x, y)$, can be viewed as the submanifold

$$M = \{(x, u_y, u, x, y)\} \subset J^1\pi = \{(u_x, u_y, u, x, y)\}.$$

– Associated relative algebroid: with $N = \{(x, y, u)\}$

$$\begin{tikzcd} \underline{\mathbb{R}^2} \arrow[r] \arrow[d] & A = \underline{\mathbb{R}^2} \arrow[d] \\ M \arrow[r, "p"'] & N \end{tikzcd}$$

$$(u_y, u, x, y) \mapsto (u, x, y)$$

$$\begin{cases} \mathrm{D}\theta^1 = \mathrm{D}\theta^2 = 0, \\ \mathrm{D}x = \theta^1, \\ \mathrm{D}y = \theta^2, \\ \mathrm{D}u = y\theta^1 + u_y\theta^2. \end{cases}$$

PDEs and relative Lie algebroids

$$\boxed{u_x = y} \qquad \Rightarrow \qquad \begin{cases} \mathrm{D}\theta^1 = \mathrm{D}\theta^2 = 0, \\ \mathrm{D}x = \theta^1, \\ \mathrm{D}y = \theta^2, \\ \mathrm{D}u = y\theta^1 + u_y\theta^2. \end{cases}$$

Free variable is u_y, so space of pointwise lifts:

$$L = \{\mathrm{D}_1 \mid \mathrm{D}_1 u_y = u_{xy}\theta^1 + u_{yy}\theta^2\}.$$

To find first prolongation:

$$0 = \mathrm{D}_1\mathrm{D}u = \theta^2 \wedge \theta^1 + (u_{xy}\theta^1 + u_{yy}\theta^2) \wedge \theta^2 = (u_{xy} - 1)\theta^1 \wedge \theta^2 \ \Rightarrow \ u_{xy} = 1.$$

So $M_1 := \{(x, u_y, u, x, y, u_{yy})\} \subset J^2\pi$ and 1st prolonged relative derivation is

$$\Rightarrow \qquad \begin{cases} \mathrm{D}_1\theta^1 = \mathrm{D}_1\theta^2 = 0, \\ \mathrm{D}_1 x = \theta^1, \\ \mathrm{D}_1 y = \theta^2, \\ \mathrm{D}_1 u = y\theta^1 + u_y\theta^2, \\ \mathrm{D}_1 u_y = \theta^1 + u_{yy}\theta^2, \end{cases} \qquad \text{(with } u_{yy} \text{ a free variable).}$$

PDEs and relative Lie algebroids

Data:

- $q : N \to X$ – submersion
- $p_k : J^k q \to J^{k-1} q$, $q_k : J^k q \to X$ – jet bundle projections
- $E \subseteq J^k q$ – submanifold (k^{th}-order PDE)

Assumption: $p_k(E) \subset J^{k-1} q$ is manifold and $q_k : E \to X$ is a submersion.

PDEs and relative Lie algebroids

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Assumption: $p_k(E) \subset J^{k-1}q$ is manifold and $q_k : E \to X$ is a submersion.

Definition

The **relative algebroid of the PDE** E is $(q_{k-1}^* TX, p_k, \mathrm{D}_E)$ where $p_k\ E \to p_k(E)$, and

$$\begin{cases} \mathrm{D}_E f = (p_k^* \mathrm{d} f)|_{\mathcal{C}}, & \text{if } f \in C^\infty(p_k(E)), \\ \mathrm{D}_E(q_{k-1}^* \alpha) = i_E^* q_k^* \mathrm{d}\alpha, & \text{if } \alpha \in \Omega^\bullet(X). \end{cases}$$

with $i_E\ E \hookrightarrow J^k q$ the inclusion and $\mathcal{C}$ the restriction of the Cartan distribution to $p_k(E)$.

PDEs and relative Lie algebroids

Let $E \subset J^k q$ be a PDE with relative algebroid $(q_{k-1}^* TX, p_k, \mathrm{D}_E)$. Then:

- germs of solutions to E are in 1-1 correspondence with germs of realizations of the relative algebroid modulo diffeomorphisms;
- E is k-integrable iff its relative algebroid is k-integrable;
- E is formally integrable iff its relative algebroid is formally integrable.
- Solutions, symmetries, reduction of E can be understood in terms of the integrations of its relative algebroid.

PDEs and relative Lie algebroids

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Conclusion

Geometric theory of PDEs $\subset$ (relative) Lie algebroid/groupoid theory

Some references

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Thank you!